Positivity cases, estimates and asymptotic expansions for condenser capacities

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POSITIVITY CASES, ESTIMATES AND ASYMPTOTIC EXPANSIONS FOR CONDENSER CAPACITIES.

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Abstract. We study positivity cases, estimates and asymptotic expansions of condenser p-capacities of points and segments in a given bounded domain, having in mind application fields, such as imaging, requiring detection and quantification of zero measure sets. We first establish estimates of capacities when the internal part of the condenser has a non-empty interior. The study of the point and its approximation by balls of small radiuses follow. Our main contribution is then to introduce equidistant condensers and to establish the positivity cases of d-dimensional condenser capacities of segments in a new way bringing out the relationship with the d-dimensional and more significantly with the $(d-1)$-dimensional condenser capacities of points. We discuss how equidistant condensers might allow to obtain by induction the posivity cases for compact submanifolds of higher dimensions. For estimation purposes, we then introduce elliptical condensers and provide an estimate and the asymptotic expansion for the condenser capacity of a segment in the harmonic case $p = 2$.

1. Condenser capacities

1.1. **Introduction.** The concept of capacity originated from the physics of electrostatic condensers. It has since then been widely extended on the mathematical side as set functions in the linear potential theory (e.g. for a review Brelot [7]) and more recently in the nonlinear potential theory (Maz'ya [17, 18], Adams & Hedberg [1], Heinonen, Kilpeläinen & Martio [12] and Turesson [27]). Many types of capacities were studied. An axiomatic theory of capacity was introduced by Choquet in the 1950's ([8] or Doob [9]).

Capacities are ubiquitous in the study of the local behavior of solutions to quasilinear partial differential equations of second order. For instance, an important feature (Serrin [22]) is that under relevant assumptions, a solution of such a PDE, defined in a given domain $\Omega \subset \mathbb{R}^d$, except on a compact subset K, can be continuously extended to a solution of the PDE in Ω, provided that K has a zero variational capacity. Letting $p > 1$, the variational p-capacity of K denoted $c_p(K)$ is defined as the minimum deviation of energy that is induced by the presence of the obstacle K. Denoting $\|u\|_{1,p}^p = \int_{\mathbb{R}^d} |u|^p + |\nabla u|^p$ for any $u \in W^{1,p}(\mathbb{R}^d)$ one defines $c_p(K) := \inf \left\{ \|u\|_{1,p}^p \ ; \ u \in C_0^\infty(\mathbb{R}^d) \text{ and } u \geq 1 \text{ in } K \right\}$. Let K be a compact submanifold of dimension k in $\mathbb{R}^d$. Then we have ([1]):

$$c_p(K) > 0 \quad \text{if and only if} \quad p > d - k. \tag{1.1}$$

Since when $p > d$ any non-empty set has a positive variational capacity, the emphasis is usually put on the case $p \leq d$ (see e.g. Attouch, Buttazzo, Michaille [4]).

In contrast, we shall focus here on the fact that, chosing appropriate parameter p, capacities take positive values on zero Lebesgue measure sets, and even on sets with codimensions ≥ 2, such as points in a 2D image or curves in a 3D image. While Hausdorff measures ([29]) enjoy similar properties, it appeared in recent years that recurring to them lead to major

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challenges, for instance when it comes to quantify lengths of curves or to detect points in imaging (see e.g. G. Aubert, L. Blanc-Féraud and R. March [5] about the implementation of the Mumford-Shah functional [20] by means of Γ-convergence [6] and more recently D. Graziani, L. Blanc-Féraud, G. Aubert [10] about the detection of points in a 2D-image).

Investigating to what extend capacities may serve as alternatives for Hausdorff measures, not only the positivity of capacities becomes an issue. So does the ability to estimate and to approximate the values of such capacities. Moreover, looking towards applications, the relevant domain is more often a bounded domain Ω. Thus one cannot a priori dismiss the subsequent need to consider the capacity of a set within a given bounded domain, instead of doing so in the whole space $\mathbb{R}^d$.

The purpose of this article is to start answering such questions. To introduce its goals and results, we first need to recall basic facts about condenser capacities. In all this article, let $p \in (1, +\infty)$, $d \in \mathbb{N}, d \geq 1$ and let a bounded domain (open connected set) $\Omega \subset \mathbb{R}^d$ and a compact subset $K \subset \Omega$. The symbol $|\cdot|$ denotes the usual euclidean norm, S^{d-1} denotes the unit sphere in $\mathbb{R}^d$ and A^{d-1} its surface area. For clarity we set $\beta := (p-d)/(p-1) \in (-\infty, 1]$.

1.2. **Condenser p-capacities.** Since a Poincaré inequality holds in $W_0^{1,p}(\Omega)$, Heinonen *et al.* [12] set

Definition 1.1. Let $W(K, \Omega) := \{u \in C_0^\infty(\Omega) : u \geq 1 \text{ in } K\}$ and define

$$C_{p,d}(K, \Omega) := \inf_{u \in W(K,\Omega)} \int_\Omega |\nabla u|^p.$$

The number $C_{p,d}(K, \Omega)$ is the condenser p-capacity of the condenser (K, Ω).

Using an approximation argument, the set $W(K, \Omega)$ can be replaced in Definition 1.1 by the larger set $W_0(K, \Omega) := \left\{u \in W_0^{1,p}(\Omega) \cap C(\Omega) : u \geq 1 \text{ in } K\right\}$. The compact K will be called the 'internal part' of the condenser and a function $u \in W_0(K, \Omega)$ an admissible function for the condenser. For simplicity, we henceforth drop the word 'condenser' and simply say 'p-capacity' instead of 'condenser p-capacity' when no confusion is possible. Similarly we drop the d of $C_{p,d}(K, \Omega)$ simply writing $C_p(K, \Omega)$ whenever no confusion is possible about the dimension of the ambient space.

Condenser capacities comply with Choquet's definition since [12],

Theorem 1.2. *The set function $K \to C_p(K, \Omega)$, where K is a compact included in the domain $\Omega \subset \mathbb{R}^d$, enjoys the following properties:*

(i) *(Monotony) If $K_1 \subset K_2 \subset \Omega$ then $C_p(K_1, \Omega) \leq C_p(K_2, \Omega)$.*
(ii) *(Monotony) If $K \subset \Omega_1 \subset \Omega_2$ then $C_p(K, \Omega_2) \leq C_p(K, \Omega_1)$.*
(iii) *(Subadditivity) If $K_1 \subset \Omega$ and $K_2 \subset \Omega$ then*

$$C_p(K_1 \cup K_2, \Omega) + C_p(K_1 \cap K_2, \Omega) \leq C_p(K_1, \Omega) + C_p(K_2, \Omega).$$

(iv) *(Descending continuity) If $(K_n)_{n\geq 0}$ is a decreasing sequence of compact subsets of Ω, that is $\Omega \supset K_0 \supset K_1 \supset \cdots \supset K_n \supset K_{n+1} \supset \cdots$ and $K := \cap_{n\geq 0} K_n$, then*

$$C_p(K, \Omega) = \lim_{n\to+\infty} \downarrow C_p(K_n, \Omega).$$

(v) *(Ascending continuity) If $(K_n)_{n\geq 0}$ is an ascending sequence of compact subsets of Ω and if $K := \cup_{n\geq 0} K_n$ is compact, then*

$$C_p(K, \Omega) = \lim_{n\to+\infty} C_p(K_n, \Omega).$$

1.3. **Goals and results of the article.** One *a priori* expects the condenser capacity of (K, Ω) to depend on the shape of K but also on its localization in Ω and on the shape and size of Ω. For instance let a point $\{x_0\} \subset \Omega \subset \mathbb{R}^2$ we shall see that $C_3(\{x_0\}, \Omega) > 0$ while $C_3(\{x_0\}, \mathbb{R}^2) = 0$. Moreover it is easy to see that $C_p(K, \Omega) = 0$ implies $c_p(K) = 0$ and accordingly $c_p(K) > 0$ implies $C_p(K, \Omega) > 0$. Then if follows from rule (1.1) that if K be a compact submanifold of dimension k, with $k > d - p$, then $C_p(K, \Omega) > 0$. But if $k \leq d - p$, is $C_p(K, \Omega)$ positive or null? Heinonen *et al.* [12] provide two positivity results which do not apply when the bounded domain Ω is given once for all.

Furthermore, while variational capacities in $\mathbb{R}^d$ were extensively studied, estimates of condenser capacities in a given domain Ω remain mostly, to our best knowledge, to be obtained. Only the capacities of spherical condensers were calculated explicitly in the literature ([12] or [18]).

Therefore the present article focuses on condenser capacities of points and segments in a given bounded domain Ω. While the case of a point may partly be asymptotically reduced to the study an isotropic p-Laplace equation, the perturbation entailed by a segment-shaped obstacle in the p-Laplace equation leads to consider a strongly anisotropic problem. Most available results regarding p-Laplace problems address the case of isolated singularities (Serrin [22, 23], Kichenassamy & Véron [13] and for a recent review Véron [28]). Anisotropic p-harmonic functions in the form $u(x) = |x|^\lambda \ \omega(x/|x|)$, where $\lambda \in \mathbb{R}$ and ω is defined on S^{d-1}, were studied for quasilinear equations with Dirichlet conditions in domains with conical boundary points (see Tolksdorf [25] and Porretta & Véron [21]). But the effect of the anisotropy induced in the p-Laplace equation by a prolate ellipsoid or a segment obstacle, has not yet been calculated.

In the preliminary section, we show how to calculate a condenser p-capacity by solving a p-Laplace equation with Dirichlet condition and we provide asymptotic bounds to the p-capacity of any condenser of which the internal part has a non-empty interior. We give a direct proof of the positivity rule for capacities of points. For approximation purposes we provide the speed of descent of the p-capacities of balls down to that of a point.

Then our main contribution is to introduce the definition of *equidistant condensers* and to implement a new method to prove the positivity rule for capacities of segments in a given bounded domain in section 3. We illustrate how the equidistant condensers method might be applied to compact submanifolds of higher dimensions. The purpose of section 4 is then to estimate condenser capacities of segments, when positive. For such estimating purposes, we introduce elliptical condensers. In the linear case $p = d = 2$, we obtain a sharp estimate and the asymtotic expansion of the condenser capacity of the segment. In the general case $p > 1, p \neq 2$, we briefly discuss how elliptic condensers might prove useful for numerical computations.

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2. Preliminary results for condenser capacities

2.1. **Estimate of p-capacity through a p-Laplace problem with Dirichlet boundary condition.** Consider the p-Laplace problem in $\Omega\backslash K$ with Dirichlet boundary condition:

$$\begin{cases} -\Delta_p(u) = 0 & \text{in } \Omega\backslash K \\ u = 1 & \text{on } \partial K \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

where Δ_p denotes the p-Laplace operator $\Delta_p(u) := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$. Assuming that both Ω and K have smooth C^1-boundaries, questions about existence, unicity and regularity of solution to Problem (2.1) are well known. According to Lindqvist [15], Problem (2.1) admits a unique solution $u \in W^{1,p}(\Omega\backslash K)$. One equivalently defines u as being the unique function that minimizes the functional $J(v) := \int_{\Omega\backslash K} |\nabla v|^p$ in the affine space $g + W_0^{1,p}(\Omega\backslash K)$, where $g \in C_0^\infty(\Omega)$ is chosen such that $g = 1$ on a neighborhood of K. In addition after Tolksdorf [26] or Wang [31], u is continuous in $\overline{\Omega\backslash K}$ (after a redefinition in a set of zero measure) and u is C^1 in $\Omega\backslash K$. In particular we have $u = 0$ on $\partial\Omega$ and $u = 1$ on ∂K pointwise.

Proposition 2.1. *Let K be a compact set of a bounded domain $\Omega \subset \mathbb{R}^d$, both with C^1-boundaries and let $u \in W^{1,p}(\Omega\backslash K) \cap C\left(\overline{\Omega\backslash K}\right)$ be the unique solution to Problem (2.1). Then*

$$C_p(K,\Omega) = \int_{\Omega\backslash K} |\nabla u|^p.$$

Proof. Let $\tilde{u}$ be the extension of u in Ω obtained by setting $\tilde{u} = 1$ in K. Clearly $\nabla\tilde{u} = 0$ in $\mathring{K}$ and $\tilde{u} \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$. Thus $\tilde{u}$ is admissible for (K,Ω). Hence $C_p(K,\Omega) \leq \int_\Omega |\nabla\tilde{u}|^p = \int_{\Omega\backslash K} |\nabla u|^p$.
Conversely, according to definition (1.1), let $(u_n)_{n\geq 0}$ a sequence $\subset W(K,\Omega)$ such that $C_p(K,\Omega) = \lim_{n\to+\infty} \int_\Omega |\nabla u_n|^p$. Define $w_n := \inf(u_n, 1)$ in Ω. It follows e.g. from [12] Theorem 1.20 that $w_n \in W_0^{1,p}(\Omega) \cap C(\Omega)$ and that $|\nabla w_n(x)| \leq |\nabla u_n(x)|$ for almost all $x \in \Omega$. In addition $u_n \geq 1$ in K implies that $w_n = 1$ in K and $\nabla w_n = 0$ in $\mathring{K}$. Thus $w_n \in W_0(K,\Omega)$ with $\int_\Omega |\nabla w_n|^p \leq \int_\Omega |\nabla u_n|^p$. Let v_n be the restriction of w_n to $\Omega\backslash K$. We check that $v_n - g \in W_0^{1,p}(\Omega\backslash K)$. Hence for all $n \geq 1$:

$$J(u) = \int_{\Omega\backslash K} |\nabla u|^p \leq \int_{\Omega\backslash K} |\nabla v_n|^p \leq \int_\Omega |\nabla w_n|^p \leq \int_\Omega |\nabla u_n|^p.$$

Letting $n \to +\infty$ yields the second required inequality. □

Let $v \in W^{1,p}(\Omega\backslash K) \cap C(\overline{\Omega\backslash K})$ such that $v = 0$ on $\partial\Omega$ and $v = 1$ on ∂K. Let $\tilde{v}$ be the extension of v in Ω obtained by setting $\tilde{v} = 1$ in K. Clearly $\tilde{v}$ is admissible for the condenser (K,Ω) in the sense of Definition 1.1. Hence by extension we say that function v is admissible for the condenser (K,Ω).

If boundaries ∂K or ∂K are not C^1, then thanks to the two monotony properties (i) and (ii) of Theorem 1.2, we shall be able to estimate $C_p(K,\Omega)$ as long as K and Ω can be properly approximated respectively by (a sequence of) some other compact and open sets with C^1-boundaries to which we may in turn apply Proposition 2.1. This approximation technique will be applied in subsection 2.3 hereafter.

2.2. **Solutions for spherical condensers and asymptotic expansions.** We shall need the explicit value of the admissible function minimizing the energy and the asymptotic expansion of the capacity when the radius of the internal ball tends towards zero. So let a point $x_0 \in \mathbb{R}^d$, two numbers $0 < \varepsilon < R$ and the concentric balls $\overline{B}_\varepsilon := \overline{B}(x_0,\varepsilon)$ and $B_R := B(x_0,R)$.

Proposition 2.2. *Denote $s_{p,d} \in W^{1,p}(B_R \backslash \overline{B}_\varepsilon)$ the unique solution to Problem (2.1) when $K = \overline{B}_\varepsilon$ and $\Omega = B_R$ and $C_p(\varepsilon, R)$ the p-capacity of the spherical condenser $\big(\overline{B}(x_0,\varepsilon), B(x_0,R)\big)$ and $r = |x - x_0|$ for $x \in B_R \backslash \overline{B}_\varepsilon$.*
If $p = d$, then for all $x \in B_R \backslash \overline{B}_\varepsilon$ we have:

$$\begin{cases} s_{p,d}(x) = [\ln(R/r)/\ln(R/\varepsilon)]\,, \\ |\nabla s_{p,d}(x)| = [r \ln(R/\varepsilon)]^{-1}\,, \\ C_p(\varepsilon,R) = A^{d-1}\,[\ln(R/\varepsilon)]^{1-p}\,, \end{cases}$$

and for $\varepsilon > 0$ small enough:

$$C_p(\varepsilon,R) = A^{d-1}\,[-\ln\varepsilon]^{1-p}\,[1 + (p-1)(\ln R/\ln\varepsilon) + o\,(1/\ln\varepsilon)]\,.$$

If $p \neq d$, then for all $x \in B_R \backslash \overline{B}_\varepsilon$ we have:

$$\begin{cases} s_{p,d}(x) = (R^\beta - r^\beta)/(R^\beta - \varepsilon^\beta), \\ |\nabla s_{p,d}(x)| = \left|\frac{\beta}{R^\beta - \varepsilon^\beta}\right|\, r^{\beta-1}, \\ C_p(\varepsilon,R) = A^{d-1}|\beta|^{p-1}\left|R^\beta - \varepsilon^\beta\right|^{1-p}\,, \end{cases}$$

and for $\varepsilon > 0$ small enough

$$\begin{cases} C_p(\varepsilon,R) = A^{d-1}\beta^{p-1}R^{d-p}\left[1 + (p-1)\,(\varepsilon/R)^\beta + o\left(\varepsilon^\beta\right)\right] & \text{if } p > d, \\ C_p(\varepsilon,R) = A^{d-1}\,(-\beta)^{p-1}\,\varepsilon^{d-p}\left[1 + (p-1)\,(\varepsilon/R)^{-\beta} + o\left(\varepsilon^{-\beta}\right)\right] & \text{if } p < d. \end{cases}$$

The proof is obtained solving Problem (2.1) in spherical coordinates and then applying Proposition 2.1. Asymptotic expansions easily follow.

2.3. **Internal parts with non-empty interior and asymptotic expansions.** Thanks to the descending continuity property (iv) of Theorem 1.2, one can approximate the capacity of a condenser of which the internal part has an empty interior, by capacities of condensers of which the internal parts have non-empty interiors with sizes tending towards zero. Hence it is useful to provide asymptotic inequalities of capacities for the latter type of condensers (see figure (1)).

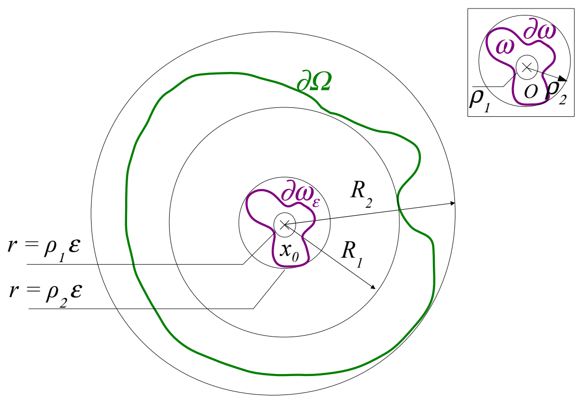

FIGURE 1. An internal part $\overline{\omega}_\varepsilon$ with a non-empty interior in a bounded domain Ω.

Let a point $x_0 \in \Omega$, $R_1 := \sup\{R > 0; B(x_0,R) \subset \Omega\} > 0$ and $R_2 := \inf\{R > 0; \Omega \subset B(x_0,R)\}$. Let a non-empty bounded domain $\omega \subset \mathbb{R}^d$ such that $0 \in \omega$ and the two numbers $\rho_1 := \sup\{\rho > 0; B(x_0,\rho) \subset \omega\}$ and $\rho_2 := \inf\{\rho > 0; \omega \subset B(x_0,\rho)\}$. Lastly set $\omega_\varepsilon := x_0 + \varepsilon \cdot \omega \subset B(x_0,R_1)$ for ε small enough and consider the condenser $(\overline{\omega}_\varepsilon, \Omega)$.

Proposition 2.3. *The following asymptotic inequalities hold.*
If $p = d$, then:

$$-A^{d-1}(p-1)\ln(R_2/\rho_1)\,[-\ln\varepsilon]^{-p} + o\left([\ln\varepsilon]^{-p}\right)$$

$$\leq \quad C_p(\overline{\omega}_\varepsilon,\Omega) - A^{d-1}\,[-\ln\varepsilon]^{1-p} \quad \leq$$

$$-A^{d-1}(p-1)\ln(R_1/\rho_2)\,[-\ln\varepsilon]^{-p} + o\left([\ln\varepsilon]^{-p}\right).$$

If $p > d$, then:

$$A^{d-1}\beta^{p-1}R_2^{d-p}\left[1 + (p-1)\,(\rho_1\,\varepsilon/R_2)^{\beta} + o\left(\varepsilon^{\beta}\right)\right]$$

$$\leq \quad C_p(\overline{\omega}_\varepsilon,\Omega) \quad \leq$$

$$A^{d-1}\beta^{p-1}R_1^{d-p}\left[1 + (p-1)\,(\rho_2\,\varepsilon/R_1)^{\beta} + o\left(\varepsilon^{\beta}\right)\right].$$

If $p < d$, then:

$$A^{d-1}\,(-\beta)^{p-1}\,(\rho_1\,\varepsilon)^{d-p}\left[1 + (p-1)\,(\rho_1\,\varepsilon/R_2)^{-\beta} + o\left(\varepsilon^{-\beta}\right)\right]$$

$$\leq \quad C_p(\overline{\omega}_\varepsilon,\Omega) \quad \leq$$

$$A^{d-1}\,(-\beta)^{p-1}\,(\rho_2\,\varepsilon)^{d-p}\left[1 + (p-1)\,(\rho_2\,\varepsilon/R_1)^{-\beta} + o\left(\varepsilon^{-\beta}\right)\right].$$

The proof is obtained noticing that after the monotony properties (i) and (ii) of Theorem 1.2, for any positive real numbers ρ', ρ'', R' and R'' such that $B(x_0,\rho'\varepsilon) \subset \omega_\varepsilon \subset B(x_0,\rho''\varepsilon) \subset B(x_0,R') \subset \Omega \subset B(x_0,R'')$, we have:

$$C_p(\rho'\varepsilon,R'') \leq C_p(\overline{B}_{\rho'\varepsilon},\Omega) \leq C_p(\overline{\omega}_\varepsilon,\Omega) \leq C_p(\overline{B}_{\rho''\varepsilon},\Omega) \leq C_p(\rho''\varepsilon,R').$$

Then applying formulae stated in Proposition 2.2 completes the proof. No assumptions at all are required about the smoothness of boundaries $\partial\overline{\omega}$ and $\partial\Omega$.

Remark 2.4. The expansions stated in Proposition 2.3 are actually topological expansions (see Masmoudi [16], Amstutz [2], Maz'ya, S. Nazarov, B. Plamenevskij [19], Sokolowski *et al.* [24] and subsequent articles). Proposition 2.3 provides the first available topological expansions in the case of a nonlinear partial differential operator.

If $p = d$, then $C_p(\overline{\omega}_\varepsilon,\Omega) = A^{d-1}\,[-\ln\varepsilon]^{1-p} + o\Big([-\ln\varepsilon]^{1-p}\Big)$. The topological gradient equals A^{d-1}. It is constant in Ω. It does not depend on the shape of the compact $\overline{\omega}$ nor on that of the domain Ω.

If $p < d$ and if ω is the unit ball, then $C_p(\overline{B}_\varepsilon,\Omega) = A^{d-1}\,(-\beta)^{p-1}\,\varepsilon^{d-p} + o\left(\varepsilon^{d-p}\right)$. The topological gradient equals $A^{d-1}\,(-\beta)^{p-1}$. It is constant in Ω. It does not depend on the shape of the domain Ω.

In the linear case $p = 2$, in 2 or 3 dimensions, the results hereabove yield the topological expansions previously proved for the Laplace operator with Dirichlet boundary condition by Guillaume & Idris [11].

Remark 2.5. In such an asymptotic approach, it is standard to change the scale, dividing all distances by ε. The internal part then becomes the unit set $\overline{\omega}$ while the outer boundary $\partial\Omega$ is sent to infinity when $\varepsilon \to 0$. We check that the outer boundary $\partial\Omega$, through parameters R_1 and R_2 do not impact the main term of the asymptotic expansion when $p \leq d$. In contrast when $p > d$, the shape of $\partial\Omega$ determines the main term of the expansion. This case exemplifies a major difference between condenser capacities in Ω and variational capacities in $\mathbb{R}^d$. It follows from the intuitive idea that the higher p, the slower the spatial diffusion process.

2.4. **Cases of positivity and estimates of p-capacities of a point.**

Theorem 2.6. *Let x_0 be a point of a bounded domain $\Omega \subset \mathbb{R}^d$. The following positivity rule holds:*

$$C_p(\{x_0\}, \Omega) > 0 \quad \textit{if and only if} \quad p > d.$$

Moreover, if $p > d$, then:

$$A^{d-1}\beta^{p-1}R_2^{d-p} \leq C_p(\{x_0\}, \Omega) \leq A^{d-1}\beta^{p-1}R_1^{d-p} \tag{2.2}$$

where $R_1 := \sup\{R > 0; B(x_0, R) \subset \Omega\}$ and $R_2 := \inf\{R > 0; \Omega \subset B(x_0, R)\}$. In particular, if $p > d$ and if $\Omega = B(x_0, R)$, then we have

$$C_p(\{x_0\}, B_R) = A^{d-1}\beta^{p-1}R^{d-p}$$

Theorem 2.6 follows from Proposition 2.3 combined with the descending continuity property (iv) of Theorem 1.2. In connection with Remark 2.5, note that when $p > d$, we have $C_p(\{x_0\}, B_R) > 0$ while $C_p(\{x_0\}, \mathbb{R}^d) = 0$.

If one wishes to obtain an estimate of the capacity of a point better than the one provided by inequalities (2.2), one may rely on the descending property (iv) of Theorem 1.2 and compute numerically the capacity of a ball with a radius r small enough. How small should be this radius depending on the required precision for the value of the capacity of the point? The following proposition answers this question.

Proposition 2.7. *If $p > d$, for $0 < r < R$, we have*

$$C_p(\overline{B}(x_0, r), B(x_0, R)) - C_p(\{x_0\}, B(x_0, R)) = O\left(r^\beta\right) \tag{2.3}$$

Proposition 2.7 follows from the expansion stated in Proposition 2.2 in the case $p > d$. When $d \geq 2$, since $0 < \beta < 1$, the speed of convergence to zero of $O\left(r^\beta\right)$ is slow when $r \to 0$.

3. Equidistant condensers. Cases of positivity for condenser p-capacities of segments

We provide in this section the comprehensive positivity rule for condenser p-capacities of segments, by means of a new method.

The natural try is to start from Proposition 2.1 and then to apply the descending property (iv) of Theorem 1.2. But as mentioned previously, the anisotropy induced by a prolate ellipsoid or by a segment obstacle in the p-Laplace equation remains uncalculated. Moreover while the definition of a condenser capacity allows to obtain upper bounds by considering energies of admissible functions, obtaining lower bounds to a capacity is a more difficult task. For these reasons, we introduce a new type of condensers, called *equidistant condensers*. Equidistant condensers are defined in order to enable a meticulous process of comparison with admissible functions of some other appropriately chosen spherical condensers. Our approach provides a lower bound to the p-capacity of a segment by means of comparison with capacities of points in dimensions d and $d-1$. Upper bounds will be obtained by extending to an equidistant condenser the solutions of two appropriately chosen spherical condensers. Therefore we conclude on the cases of positivity for condenser p-capacities of segments, depending on p and d. Lastly, we illustrate how our method based on equidistant condensers might be extended by induction reasoning to establish positivity rules for condenser capacities when the internal part is a compact submanifold of higher dimensions.

3.1. **Equidistant condensers.** Recall K is a compact subset of the bounded domain $\Omega \subset \mathbb{R}^d$. For $x \in \mathbb{R}^d$, we denote the distance $d(x, K) = \inf\{|y - x|; y \in K\}$

Definition 3.1. Let $0 < \eta < R$. Let the compact $K_\eta := \left\{x \in \mathbb{R}^d \mid d(x, K) \leq \eta\right\}$ and the bounded domain $\Omega_R := \left\{x \in \mathbb{R}^d \mid d(x, K) < R\right\}$. We say that (K_η, Ω_R) is an equidistant condenser derived from the compact K.

In all section 3, let $S_\varepsilon \subset \mathbb{R}^d$ $(d \geq 2)$, be a (closed) segment of length $\varepsilon > 0$ and centered on a point x_0. Let $0 < \eta < R$ and consider the equidistant condenser (K_η, Ω_R) derived from the segment S_ε (figure 2).

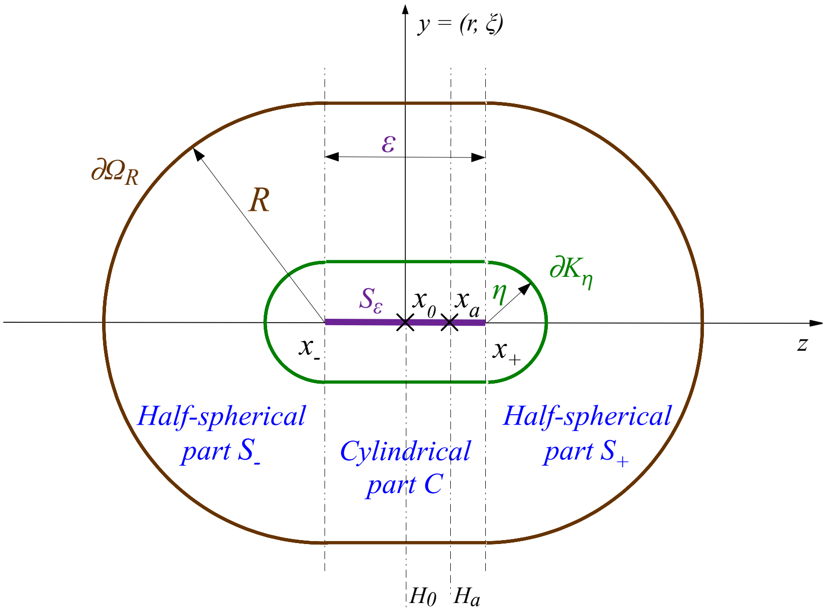

FIGURE 2. An equidistant condenser (K_η, Ω_R).

Some notations are useful. Let z be an axis passing through the point x_0 and parallel to the segment S_ε. Due to the symmetry of revolution of the condenser (K_η, Ω_R) around the z-axis, it is convenient to use the cylindrical coordinates $x = (z, y) = (z, r, \xi)$, with $z \in \mathbb{R}$, $y = r\xi \in \mathbb{R}^{d-1}$, $r \geq 0$ and $\xi \in S^{d-2}$. Let x_- (resp. x_+) the endpoint of the segment S_ε, of cylindrical coordinates $(z = -\varepsilon/2, r = 0)$ (resp. $(z = \varepsilon/2, r = 0)$).

Let $C := \{x \in \Omega_R \backslash K_\eta \, ; |z| < \varepsilon/2\}$ be the open cylindrical subset of $\Omega_R \backslash K_\eta$ and $S_\pm := \{x \in \Omega_R \backslash K_\eta \, ; \pm z > \varepsilon/2\}$ the two open half-spherical subsets of $\Omega_R \backslash K_\eta$ and $S := S_- \cup S_+$. So that $(\Omega_R \backslash K_\eta) \setminus (C \cup S)$ is of zero Lebesgue measure.

We denote $u \in W^{1,p}(\Omega_R \backslash K_\eta) \cap C\left(\overline{\Omega_R \backslash K_\eta}\right) \cap C^1\left(\Omega_R \backslash K_\eta\right)$ the unique solution to Problem (2.1) when $K = K_\eta$ and $\Omega = \Omega_R$. After Proposition 2.1, the p-capacity of the condenser (K_η, Ω_R) is $C_{p,d}\left(K_\eta, \Omega_R\right) = \int_{C \cup S} |\nabla u|^p \, dx$ where ∇ denotes the gradient operator in $\mathbb{R}^d$ and dx the Lebesgue measure in $\mathbb{R}^d$. Moreover S_ε is invariant by the orthogonal symmetry $(z, y) \mapsto (-z, y)$ relative to $H_0 := \{z = 0\}$. Thus the condenser (K_η, Ω_R) enjoys the same symmetry and so does u due to uniqueness of the solution to Problem (2.1).

Recall $s_{p,d}$ denotes the admissible function minimizing the energy of the d-dimensional spherical condenser $(\overline{B}(x_0, \eta), B(x_0, R))$ and we have $C_{p,d}(\eta, R) = \int_{B(x_0,R) \backslash B(x_0,\eta)} |\nabla s_{p,d}|^p \, dx$. The values of $s_{p,d}$ and $C_{p,d}(\eta, R)$ were provided in section 2.

Lastly for any $a \in [-\varepsilon/2, \varepsilon/2]$, let H_a be the affine hyperplane $\{z = a\}$ and x_a the intersection between H_a and the z-axis. It is pivotal to note that $(K_\eta \cap H_a, \Omega_R \cap H_a)$ is a $(d-1)$-dimensional spherical condenser. The admissible function minimizing the energy of this condenser is denoted $s_{p,d-1}$ and we have $C_{p,d-1}(\eta, R) = \int_{B_{d-1}(x_a,R) \backslash B_{d-1}(x_a,\eta)} |\nabla_y \, s_{p,d-1}|^p \, dy$, where ∇_y denotes the gradient operator in $\mathbb{R}^{d-1}$ and dy the Lebesgue measure in $\mathbb{R}^{d-1}$.

3.2. **A lower-bound to the p-capacity of a segment.**

Proposition 3.2. *With the previous notations, the p-capacity of the equidistant condenser* (K_η, Ω_R) *admits the following lower-bound*

$$C_{p,d}\left(K_\eta, \Omega_R\right) \geq C_{p,d}(\eta, R) + \varepsilon\, C_{p,d-1}(\eta, R). \tag{3.1}$$

Proof. Since $C_{p,d}\left(K_\eta, \Omega_R\right) = \int_C |\nabla u|^p\, dx + \int_S |\nabla u|^p\, dx$, we estimate separetaly each integral. In the cylindrical subset C, for any $a \in (-\varepsilon/2, \varepsilon/2)$, let w_a be the restriction of u to $H_a \cap \left(\overline{\Omega_R \backslash K_\eta}\right)$, that is $w_a(y) = u(a, y)$ for all $y \in \mathbb{R}^{d-1}$, $\eta \leq |y| \leq R$. Due to the regularity of function u, w_a is well-defined pointwise, continuous in $H_a \cap \overline{\Omega_R \backslash K_\eta}$ and w_a admits a classical gradient in $H_a \cap (\Omega_R \backslash K_\eta)$. Since u is admissible for the condenser (K_η, Ω_R), $|y| = \eta$ implies $w_a(y) = u(a, y) = 1$ and $|y| = R$ implies $w_a(y) = u(a, y) = 0$.

Moreover for all $y \in \mathbb{R}^{d-1}$, $\eta < |y| < R$ we have:

$$|\nabla_y w_a(y)| = |\nabla_y u(a, y)| \leq \Big[\ |\nabla_y u(a, y)|^2 + |\partial_z u(a, y)|^2\ \Big]^{1/2} = |\nabla u(a, y)|\,.$$

For a given $a \in (-\varepsilon/2, \varepsilon/2)$, if $\int_{H_a \cap (\Omega_R \backslash K_\eta)} |\nabla_y w_a(y)|^p\, dy\ < +\infty$, then w_a is admissible to the $(d-1)$-dimensional condenser $(\overline{B}_{d-1}(x_a, \eta), B_{d-1}(x_a, R))$. Thus:

$$C_{p,d-1}(\eta, R) \leq \int_{H_a \cap (\Omega_R \backslash K_\eta)} |\nabla_y\ w_a(y)|^p\ dy \leq \int_{H_a \cap (\Omega_R \backslash K_\eta)} |\nabla\ u(a, y)|^p\ dy. \tag{3.2}$$

If $\int_{H_a \cap (\Omega_R \backslash K_\eta)} |\nabla_y w_a(y)|^p\, dy\ = +\infty$, inequality (3.2) obviously holds again. Integrating inequality (3.2) for $a \in (-\varepsilon/2, \varepsilon/2)$, we obtain:

$$\varepsilon\, C_{p,d-1}(\eta, R) \leq \int_C |\nabla\ u(x)|^p\ dx. \tag{3.3}$$

Let v be the function defined in $\overline{B}(x_0, R) \backslash B(x_0, \eta)$ which inherits the values taken by u in the two half-spherical subsets $S_\pm$. More precisely, for all $x \in \mathbb{R}^d$, $\eta \leq |x - x_0| \leq R$, we define

$$\begin{cases} v(x) := u(x_+ + x - x_0) & \text{if } z(x - x_0) \geq 0, \\ v(x) := u(x_- + x - x_0) & \text{if } z(x - x_0) < 0. \end{cases}$$

Since u is continuous in $\overline{\Omega_R \backslash K_\eta}$ and symmetric relatively to the hyperplane H_0, it follows that v is continuous in $\overline{B}(x_0, R) \backslash B(x_0, \eta)$. Similarly $u \in L^p\left(\Omega_R \backslash K_\eta\right)$ implies that $v \in L^p\left(B(x_0, R) \backslash \overline{B}(x_0, \eta)\right)$.

For any $x \in \left(B(x_0, R) \backslash \overline{B}(x_0, \eta)\right) \cap \{z \neq 0\}$ we have

$$\begin{cases} \nabla v(x) = \nabla u(x_+ + x - x_0) & \text{if } z(x - x_0) > 0, \\ \nabla v(x) = \nabla u(x_- + x - x_0) & \text{if } z(x - x_0) < 0. \end{cases}$$

Thus $\nabla u \in L^p(\Omega_R \backslash K_\eta)$ entails $\nabla v \in L^p\left(\left(B(x_0, R) \backslash \overline{B}(x_0, \eta)\right) \cap \{z > 0\}\right)$ and similarly $\nabla v \in L^p\left(\left(B(x_0, R) \backslash \overline{B}(x_0, \eta)\right) \cap \{z < 0\}\right)$. Moreover, since v is continuous in $\overline{B}(x_0, R) \backslash B(x_0, \eta)$ and thus has no jump accross $\{z = 0\}$, the results about distribution derivatives (e.g. [30]) entail that the distribution ∇v defined in the domain $\left(B(x_0, R) \backslash \overline{B}(x_0, \eta)\right)$ can be identified to the vector field $\{\nabla v\}$ defined in $\left(B(x_0, R) \backslash \overline{B}(x_0, \eta)\right) \cap \{z \neq 0\}$. Hence $\nabla v \in L^p\left(B(x_0, R) \backslash \overline{B}(x_0, \eta)\right)$.

Recall u is admissible for the condenser (K_η, Ω_R). Thus we have $v(x) = 1$ for all $x \in \mathbb{R}^d$, $|x - x_0| = \eta$ and $v(x) = 0$ for all $x \in \mathbb{R}^d$, $|x - x_0| = R$.
Therefore v is an admissible function for the condenser $(\overline{B}(x_0, \eta), B(x_0, R))$ and it follows that:

$$C_{p,d}(\eta, R) \leq \int_{B(x_0,R) \backslash B(x_0,\eta)} |\nabla\ v(x)|^p\ dx = \int_S |\nabla\ u(x)|^p\ dx. \tag{3.4}$$

Summing inequalities (3.3) and (3.4) yields the claimed result. □

Thanks to equidistant condensers, we can now state the following lower-bound to the condenser p-capacity of a segment. Recall $C_{p,d}(\{x_0\}, B_R)$ (resp. $C_{p,d-1}(\{x_0\}, B_R)$ denotes the p-capacity of the point $\{x_0\}$ in the d-dimensional ball $B(x_0, R)$ (resp. the p-capacity of the point $\{x_0\}$ in the $(d-1)$-dimensional ball $B_{d-1}(x_0, R)$).

Theorem 3.3. *Let Ω be a bounded domain of $\mathbb{R}^d$ and $x_0 \in \Omega$. Let $R := \sup\{|y - x_0| \,; y \in \Omega\} \in (0, +\infty)$. Let S_ε be a (closed) segment centered on the point x_0 and of length $\varepsilon > 0$ such that $S_\varepsilon \subset \Omega$. Then the following lower-bound holds:*

$$C_{p,d}(S_\varepsilon, \Omega) \;\geq\; C_{p,d}(\{x_0\}, B_R) + \varepsilon\, C_{p,d-1}(\{x_0\}, B_R). \tag{3.5}$$

Proof. For any $\lambda > 0$ and any η, $0 < \eta < R$, inequality (3.1) of Proposition 3.2, applied to radiuses η and $R + \lambda$, reads:

$$C_{p,d}(\eta, R + \lambda) + \varepsilon\, C_{p,d-1}(\eta, R + \lambda) \leq C_{p,d}\left(K_\eta, \Omega_{R+\lambda}\right). \tag{3.6}$$

Three decreasing sequences of compacts are involved as follows:

$$\cap_{\eta>0}\ \overline{B}(x_0, \eta) = \{x_0\}\,,\ \cap_{\eta>0}\ \overline{B}_{d-1}(x_0, \eta) = \{x_0\} \ \text{ and } \ \cap_{\eta>0}\ K_\eta = S_\varepsilon.$$

The continuity property (iv) of Theorem 1.2 hence implies that:

$$\begin{cases} \lim\limits_{\eta\to 0} C_{p,d}(\overline{B}(x_0, \eta), B(x_0, R+\lambda)) = C_{p,d}(\{x_0\}, B(x_0, R+\lambda)) \\ \lim\limits_{\eta\to 0} C_{p,d-1}(\overline{B}(x_0, \eta), B(x_0, R+\lambda)) = C_{p,d-1}(\{x_0\}, B(x_0, R+\lambda)) \\ \lim\limits_{\eta\to 0} C_{p,d}\left(K_\eta, \Omega_{R+\lambda}\right) = C_{p,d}(S_\varepsilon, \Omega_{R+\lambda}). \end{cases}$$

Therefore passing to the limit when $\eta \to 0$ in inequality (3.6) yields

$$C_{p,d}(\{x_0\}, B_{R+\lambda}) + \varepsilon\, C_{p,d-1}(\{x_0\}, B_{R+\lambda}) \leq C_{p,d}\left(S_\varepsilon, \Omega_{R+\lambda}\right). \tag{3.7}$$

Moreover the inclusions $S_\varepsilon \subset \Omega \subset B(x_0, R + \lambda) \subset \Omega_{R+\lambda}$ hold. Hence the monotony property (ii) of Theorem 1.2 implies that

$$C_{p,d}(S_\varepsilon, \Omega_{R+\lambda}) \leq C_{p,d}(S_\varepsilon, B(x_0, R+\lambda)) \leq C_{p,d}(S_\varepsilon, \Omega). \tag{3.8}$$

Gathering inequalities (3.7) and (3.8) entails

$$C_{p,d}(\{x_0\}, B_{R+\lambda}) + \varepsilon\, C_{p,d-1}(\{x_0\}, B_{R+\lambda}) \leq C_{p,d}(S_\varepsilon, \Omega).$$

Lastly it follows from Theorem 2.6 that the mappings $R > 0 \mapsto C_{p,d}(\{x_0\}, B_R)$ and $R \mapsto C_{p,d-1}(\{x_0\}, B_R)$ are continuous. Hence letting λ tend towards 0 yields the claimed inequality. $\square$

Remark 3.4. The lower-bound of Theorem 3.3 is worth interpreting. Recall from section 2 that the capacity of point $\{x_0\}$ in a bounded ball of $\mathbb{R}^d$ is positive if and only if $p > d$. Accordingly three cases are to be considered:

- If $d - 1 < p \leq d$, the point has a null p-capacity in dimension d but a positive p-capacity in dimension $d - 1$. The inequality reads:

$$\varepsilon\, C_{p,d-1}(\{x_0\}, B_R) \leq C_{p,d}(S_\varepsilon, \Omega)$$

  In particular, $C_{p,d}(S_\varepsilon, \Omega) > 0$.
- If $p > d$, both capacities $C_{p,d}(\{x_0\}, B_R)$ and $C_{p,d-1}(\{x_0\}, B_R)$ are positive. Then again $C_{p,d}(S_\varepsilon, \Omega) > 0$.
- If $p \leq d - 1$, both capacities $C_{p,d}(\{x_0\}, B_R)$ and $C_{p,d-1}(\{x_0\}, B_R)$ are null.

Thus we can state the first part of the searched positivity rule for condenser capacities of segments.

Corollary 3.5. *Let S_ε be a segment of length $\varepsilon > 0$ included in a bounded domain $\Omega \subset \mathbb{R}^d$. If $p > d - 1$ then $C_{p,d}(S_\varepsilon, \Omega) > 0$.*

3.3. **Cases of nullity of the condenser p-capacity of a segment in a bounded domain.**

Proposition 3.6. *Let $S_\varepsilon \subset \Omega$ be a segment of length $\varepsilon > 0$ centered on a point x_0. If $p \leq d-1$, then the condenser p-capacity of the segment S_ε in the domain Ω is null, that is $C_{p,d}(S_\varepsilon, \Omega) = 0$.*

Proof. Let $\Omega^c := \mathbb{R}^d \backslash \Omega$. Since Ω is bounded there exists $M > 0$ such that $\Omega \subset B(x_0, M)$. Then S_ε and $\Omega^c \cap \overline{B}(x_0, M)$ are compacts such that $S_\varepsilon \cap \left(\Omega^c \cap \overline{B}(x_0, M)\right) = \emptyset$. Therefore due to the continuity of the distance, there exist $x_a \in S_\varepsilon$ and $x_b \in \Omega^c \cap \overline{B}(x_0, M)$ such that:

$$|x_a - x_b| = \min\left\{|x_1 - x_2|\,; x_1 \in S_\varepsilon \text{ and } x_2 \in \Omega^c \cap \overline{B}(x_0, M)\right\} > 0.$$

Let $R := |a-b|/2$. We have $S_\varepsilon \subset \Omega_R \subset \Omega$ thus $C_{p,d}(S_\varepsilon, \Omega) \leq C_{p,d}(S_\varepsilon, \Omega_R)$. Therefore it suffices to prove that $C_{p,d}(S_\varepsilon, \Omega_R) = 0$. Moreover due to the descending continuity property (iv) of Theorem 1.2, it suffices to prove that

$$\lim_{\eta \to 0} C_p(K_\eta, \Omega_R) = 0 \tag{3.9}$$

We first prove (3.9) in the case $p < d-1$. Let the function $v : \overline{\Omega_R \backslash K_\eta} \to \mathbb{R}$ defined by:

$$\begin{cases} \text{if } x \in \overline{S}_- \cap \{z < -\varepsilon/2\} & \text{then } v(x) := s_{p,d}(\rho_-) \text{ with } \rho_- = |x - x_-|\ , \\ \text{if } x \in \overline{S}_+ \cap \{z > \varepsilon/2\} & \text{then } v(x) := s_{p,d}(\rho_+) \text{ with } \rho_+ = |x - x_+|\ , \\ \text{if } x \in \overline{C} & \text{then } v(x) := s_{p,d}(r) \text{ with } r = |y|. \end{cases}$$

It is easy to check that v is continuous in $\overline{\Omega_R \backslash K_\eta}$, that $v \in W^{1,p}(\Omega_R \backslash K_\eta)$ and that $v = 0$ on $\partial\Omega_R$ and $v = 1$ on ∂K_η. Thus v is admissible for the condenser (K_η, Ω_R). Hence

$$C_{p,d}\left(K_\eta, \Omega_R\right) \leq \int_{C \cup S} |\nabla v|^p \, dx,$$

so that it suffices to prove that $\lim\limits_{\eta \to 0} \displaystyle\int_{C \cup S} |\nabla v|^p \, dx = 0$.

By definition of $s_{p,d}$ we have $\int_S |\nabla v|^p \, dx = C_{p,d}(\eta, R)$. Since $p < d$, it follows from Theorem 2.6 that $\lim\limits_{\eta \to 0} \displaystyle\int_S |\nabla v|^p \, dx = 0$.

Furthermore an integration in cylindrical coordinates in C yields:

$$\int_C |\nabla v|^p \, dx = \varepsilon \, A^{d-2} \int_\eta^R |\partial_r s_{p,d}(r)|^p r^{d-2} \, dr$$

As $p < d-1$ after Proposition 2.2 we have

$$|\partial_r s_{p,d}(r)| = \left[-\beta/(\eta^\beta - R^\beta)\right] \, r^{\beta - 1}.$$

Hence

$$\int_\eta^R |\partial_r s_{p,d}(r)|^p r^{d-2} \, dr \; = \left[\frac{-\beta}{\eta^\beta - R^\beta}\right]^p \, \frac{\eta^{\beta-1} - R^{\beta-1}}{1-\beta}.$$

Since $\beta < 0$, when η tends towards 0, the integral is equivalent to $\frac{(-\beta)^p}{1-\beta} \; \eta^{\beta - 1 - p\beta}$ with $\beta - 1 - p\beta = d - p - 1 > 0$. It follows that

$$\lim_{\eta \to 0} \int_\eta^R |\partial_r s_{p,d}(r)|^p r^{d-2} \, dr = 0$$

and that $\lim\limits_{\eta \to 0} \displaystyle\int_C |\nabla v|^p \, dx = 0$ which completes the proof of (3.9) in the case $p < d-1$.

We then prove (3.9) in the case $p = d-1$. Let the function $w : \Omega_R \backslash K_\eta \to \mathbb{R}$ defined by:

$$\begin{cases} \text{if } x \in \overline{C} & \text{then } w(x) := s_{p,d-1}(r) \text{ with } r = |y|, \\ \text{if } x \in \overline{S}_- \cap \{z < -\varepsilon/2\} & \text{then } w(x) := s_{p,d-1}(\rho_-) \text{ with } \rho_- = |x - x_-|\,, \\ \text{if } x \in \overline{S}_+ \cap \{z > \varepsilon/2\} & \text{then } w(x) := s_{p,d-1}(\rho_+) \text{ with } \rho_+ = |x - x_+|\,. \end{cases}$$

As for function v, it is easy to check that w is an admissible function for the condenser (K_η, Ω_R). Hence

$$C_{p,d}\left(K_\eta, \Omega_R\right) \leq \int_{C \cup S} |\nabla w|^p \, dx,$$

so that it suffices to prove that $\lim_{\eta \to 0} \int_{C \cup S} |\nabla w|^p \, dx = 0$.

By definition of $s_{p,d-1}$ we have $\int_C |\nabla w|^p \, dx = \varepsilon \, C_{p,d-1}(\eta, R)$. Since $p = d-1$, recall from Theorem 2.6 that $\lim_{\eta \to 0} \int_C |\nabla w|^p \, dx = 0$.

Furthermore an integration in spherical coordinates in S yields:

$$\int_S |\nabla w|^p \, dx = A^{d-1} \int_\eta^R |\partial_\rho s_{p,d-1}(\rho)|^p \rho^{d-1} \, d\rho.$$

As $p = d-1$, the gradient reads: $|\partial_\rho s_{p,d-1}(\rho)| = \frac{1}{\ln(R/\eta)} \, \frac{1}{\rho}$.

Hence

$$\int_\eta^R |\partial_\rho s_{p,d-1}(\rho)|^p \rho^{d-1} \, d\rho = |\ln(R/\eta)|^{-p} \; (R - \eta).$$

Therefore $\lim_{\eta \to 0} \int_\eta^R |\partial_\rho s_{p,d-1}(\rho)|^p \rho^{d-1} \, d\rho = 0$ and thus $\lim_{\eta \to 0} \int_S |\nabla w|^p \, dx \; = 0$ which completes the proof of (3.9) in the case $p = d-1$. □

After Corollary 3.5 and Proposition 3.6, we can state the positivity rule for condenser p-capacities of segments.

Theorem 3.7. *The condenser p-capacity of a segment S_ε of length $\varepsilon > 0$ included in a bounded domain $\Omega \subset \mathbb{R}^d$ is positive if and only if $p > d-1$.*

For instance, the choice $d - 1 < p \leq d$, e.g. $p = 3$ in 3 dimensions, seems to be a good candidate for the detection of one dimensional singularities since segments have positive capacities while points and hopefully part of the noise have null capacity.

3.4. **Further developments with equidistant condensers.** It appeared in Theorem 3.3 that the positivity of the condenser p-capacity of a segment in a d-dimensional bounded domain follows from the positivity of the condenser p-capacity of a point in a $(d-1)$-dimensional bounded domain.

Recurring again to equistant condensers, we may think of a proof similar to the one of Proposition 3.2 in order to show that the positivity of the p-capacity of a plane rectangle in a d-dimensional bounded domain follows from the positivity of the p-capacity of a segment in a $(d-1)$-dimensional bounded domain, which happens when $p > (d-1)-1$. Such reasonings could be extended by induction to prove that the condenser p-capacity of a k-dimensional closed box in a d-dimensional bounded domain is positive as soon as $p > d - k$.

The cases of nullity for condenser capacity of a k-dimensional closed box seem to be more intricate to establish by means of equidistant condensers as the relationship between the capacity of a segment in a d-dimensional domain and the one of a point in a $(d-1)$-dimensional domain is not straightforward in the proof of Proposition 3.6.

4. Elliptical condensers. Estimates for condenser p-capacities of segments

When $p > d-1$, the next arising question is about estimating the capacity of a segment in a bounded domain. For this purpose we introduce elliptical condensers which ease recurring to elliptic coordinates. Let again a (closed) segment $S_\varepsilon \subset \mathbb{R}^d$ ($d \geq 2$), of length $\varepsilon > 0$ and centered on a point x_0. Let z be an axis passing through the point x_0 and parallel to the segment S_ε. We consider the cylindrical coordinates $(z, y) = (z, r, \xi)$, with $z \in \mathbb{R}$, $y = r\xi \in \mathbb{R}^{d-1}$, $r \geq 0$ and $\xi \in S^{d-2}$. Then we move forward to the elliptic coordinates (μ, ν, ξ) (see [14] or [32]) implicitely defined as follows for $\mu \in [0, +\infty)$, $\nu \in [0, \pi]$ and $\xi \in S^{d-2}$:

$$\begin{cases} z(\mu,\nu) & := \varepsilon/2 \; \cosh\mu\cos\nu, \\ r(\mu,\nu) & := \varepsilon/2 \; \sinh\mu\sin\nu, \\ \xi & := \xi, \end{cases} \tag{4.1}$$

so that $S_\varepsilon = \{\mu = 0, \nu \in [0, \pi]\}$.

4.1. Elliptical condensers. Looking at figure 3 we set

Definition 4.1. Let $0 < \eta < M$. Let the bounded domain

$$\Omega_M := \left\{ x = (\mu, \nu, \xi) \in \mathbb{R}^d \;;\; 0 \leq \mu < M, \nu \in [0, \pi], \xi \in S^{d-2} \right\}$$

and the compact

$$K_\eta := \left\{ x = (\mu, \nu, \xi) \in \mathbb{R}^d \;;\; 0 \leq \mu \leq \eta, \nu \in [0, \pi], \xi \in S^{d-2} \right\}.$$

We say that (K_η, Ω_M) is an elliptical condenser derived from the segment S_ε.

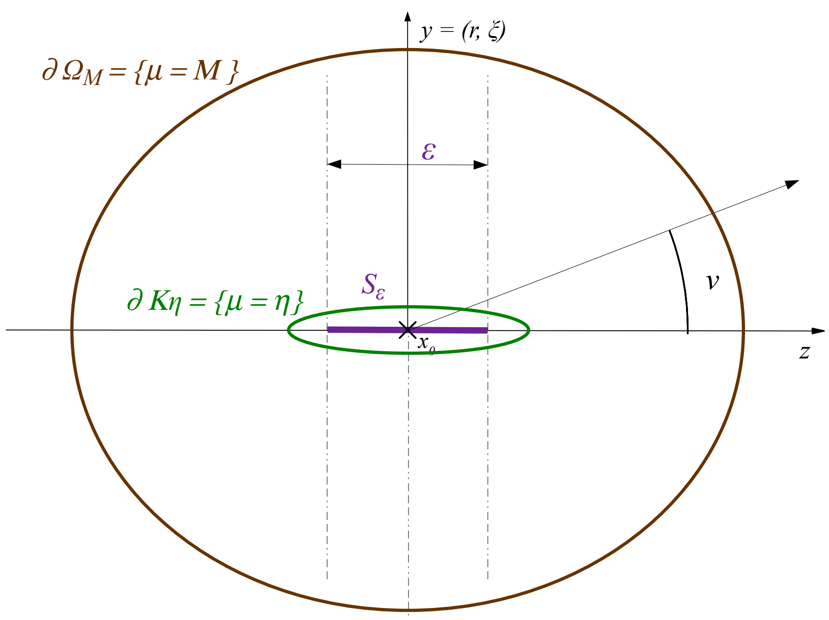

Figure 3. An elliptic condenser (K_η, Ω_M).

Obviously the inclusions $S_\varepsilon \subset K_\eta \subset \Omega_M$ hold for any $0 < \eta < M$. Moreover we have $\cap_{\eta>0} K_\eta = S_\varepsilon$. In comparison with equidistant condensers though, letting $\eta \to 0$ will not be sufficient to approximate asymptotically, when $\varepsilon \to 0$, the condenser made of the segment S_ε within a given bounded domain Ω. Indeed due to (4.1), for a given $M > 0$, $\Omega_M \to \{x_0\}$

when $\varepsilon \to 0$. So that we shall have to choose some appropriate $M(\varepsilon) \to +\infty$ to approximate a given domain Ω when letting $\varepsilon \to 0$.

Lemma 4.2. *Let $R > \varepsilon/2$ and set $M' := \ln\left(2R/\varepsilon + \sqrt{1 + 4R^2/\varepsilon^2}\right)$ and $M'' := \ln\left(2R/\varepsilon + \sqrt{-1 + 4R^2/\varepsilon^2}\right)$. Let K a compact of $\mathbb{R}^d$ such that $K \subset \Omega_{M''}$. Then we have*

$$C_p(K, \Omega_{M'}) \le C_p(K, B_R) \le C_p(K, \Omega_{M''}). \tag{4.2}$$

In particular, for any η, $0 < \eta < M''$, we have

$$C_p(K_\eta, \Omega_{M'}) \le C_p(K_\eta, B_R) \le C_p(K_\eta, \Omega_{M''}). \tag{4.3}$$

and

$$C_p(S_\varepsilon, \Omega_{M'}) \le C_p(S_\varepsilon, B_R) \le C_p(S_\varepsilon, \Omega_{M''}) \tag{4.4}$$

Proof. It follows from (4.1) that the inclusions $B_{\frac{\varepsilon}{2}\sinh M} \subset \Omega_M \subset B_{\frac{\varepsilon}{2}\cosh M}$ hold for any $M > 0$. Note that $R = \frac{\varepsilon}{2}\,\sinh M' = \frac{\varepsilon}{2}\,\cosh M''$. Hence letting $M = M'$ and $M = M''$, we obtain $\Omega_{M''} \subset B_R \subset \Omega_{M'}$. Then the monotony property (ii) of Theorem 1.2 implies

$$C_p(K, \Omega_{M'}) \le C_p(K, B_R) \le C_p(K, \Omega_{M''}) \tag{4.5}$$

which, considering $K = K_\eta$ or $K = S_\varepsilon$, entails both (4.3) and (4.4). $\square$

4.2. **The condenser 2-capacity of a segment.** In the harmonic case $p = 2$, the condenser capacity of a segment is positive in a bounded domain of $\mathbb{R}^2$. In higher dimensions, the capacity is null.

Proposition 4.3. *Let $0 < \varepsilon/2 < R$. Let S_ε a segment centered on a point x_0 and of length ε and let $B_R = B(x_0, R)$ be both subsets of $\mathbb{R}^2$. Then the following inequalities hold:*

$$\frac{2\pi}{\ln\left(2R/\varepsilon + \sqrt{1 + 4R^2/\varepsilon^2}\right)} \le C_2(S_\varepsilon, B_R) \le \frac{2\pi}{\ln\left(2R/\varepsilon + \sqrt{-1 + 4R^2/\varepsilon^2}\right)}.$$

Proof. We compute $C_p(K_\eta, \Omega_M)$ applying Proposition 2.1. Due to the symmetry of revolution relatively to the z-axis, the searched solution does not depend upon ξ. Thus in elliptic coordinates, the Laplace operator is given by:

$$\Delta u(\mu, \nu) = (4/\varepsilon^2)\left(\partial_{\mu\mu} u + \partial_{\nu\nu} u\right) / \left(\sinh^2 \mu + \sin^2 \nu\right).$$

Problem (2.1) reads $\partial_{\mu\mu} u + \partial_{\nu\nu} u = 0$ in $\Omega_M \backslash K_\eta$ with Dirichlet boundary condition $u(\eta, \nu) = 1$ and $u(M, \nu) = 0$ for all $\nu \in [0, \pi]$. The separation of variables provides: $u(\mu, \nu) = (M - \mu)/(M - \eta)$. Then

$$|\nabla u|^2 = \frac{4}{\varepsilon^2\left(\sinh^2 \mu + \sin^2 \nu\right)}\;\frac{1}{(M - \eta)^2}$$

Since $|\det D(z, r, \xi)/D(\mu, \nu, \xi)| = (\varepsilon/2)^2\left(\sinh^2 \mu + \sin^2 \nu\right)$, the change of variables leads to

$$C_2(K_\eta, \Omega_M) = \int_{\Omega_M \backslash K_\eta} |\nabla u|^2 = 2\pi/(M - \eta).$$

The descending continuity (iv) of Theorem 1.2 gives $C_2(S_\varepsilon, \Omega_M) = 2\pi/M$. Applying the latter equality for both $M = M'$ and $M = M''$ and Lemma 4.2 with (4.4) yields the claimed inequalities. $\square$

Corollary 4.4. *Let $S_\varepsilon \subset \mathbb{R}^2$ be a segment centered on a point x_0 and of length $\varepsilon > 0$. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain such that $x_0 \in \Omega$. Then for ε small enough:*

$$C_2(S_\varepsilon, \Omega) = \frac{2\pi}{-\ln \varepsilon} + o\left(\frac{1}{\ln \varepsilon}\right).$$

The expansion of Corollary 4.4 follows from inequalities of Proposition 4.3. The first-order expansion of $C_2(S_\varepsilon, \Omega)$ does neither depend on x_0 nor on the shape of Ω. For the study of the perturbation of the Laplace equation in 2D by a Neumann homogeneous boundary condition on a segment, see [3].

Note that, for any $R > \varepsilon/2$, the monotony property (ii) of Theorem 1.2 and the upper-bound of Proposition 4.3 imply

$$0 \leq C_2(S_\varepsilon, \mathbb{R}^2) \leq C_2(S_\varepsilon, B_R) \leq 2\pi/\ln\left(2R/\varepsilon + \sqrt{-1 + 4R^2/\varepsilon^2}\right).$$

Letting $R \to +\infty$ yields $C_2(S_\varepsilon, \mathbb{R}^2) = 0$.

4.3. **Further developments with elliptical condensers.** There are some hints that elliptical condensers might prove useful to estimate numerically the p-capacity of a segment S_ε in a given domain Ω, approximating S_ε by an ellipsoid K_η, for η 'small enough'. Indeed chosing a convenient geometry and adequate coordinates before discretization is obviously crucial. An equidistant condenser would be cumbersome in the sense that the p-Laplace operator would be discretized in d-dimensional spherical coordinates in the sets $S_\pm$ and in in $(d-1)$-dimensional cylindrical coordinates in the sets C leaving unsolved a delicate transition between $S_\pm$ and C.

Elliptic coordinates seem advisable in the sense that they continuously account for the transition from the d-dimensional equation located at the two end points of S_ε, to the roughly speaking $(d-1)$-dimensional equation located in the cylindrical part of the condenser. Furthermore, solutions and integrals are to be computed on the rectangle $\mathcal{R} := [\eta, M] \times [0, \pi]$. For instance, denoting the weights

$$E(\mu, \nu) := \frac{(\sinh\mu \sin\nu)^{d-2}}{\left(\sinh^2\mu + \sin^2\nu\right)^{\frac{p-2}{2}}}.$$

for any $\mu > 0$ and $0 \leq \nu \leq \pi$, one may obtain $C_p(K_\eta, \Omega_M)$ computing the following minimization problem

$$A^{d-2}\left(\frac{\varepsilon}{2}\right)^{d-p} \inf \int_{\mathcal{R}} E(\mu,\nu)\,|\nabla v(\mu,\nu)|^p \,\mathrm{d}\mu\,\mathrm{d}\nu,$$

the infimum being searched among admissible functions $v : \mathcal{R} \to \mathbb{R}$, such that $v(\eta, \nu) = 1$ and $v(M, \nu) = 0$ for all $\nu \in [0, \pi]$.

Such ideas remain to be tested numerically.

5. Conclusion and future prospects

In this paper, we first recall the definition and basic properties of condenser p-capacities of compact sets in bounded domains, emphasizing the differences with the usual variational capacities in $\mathbb{R}^d$ and mentioning why condenser capacities are likely to prove useful as substitutes for Hausdorff measures in application fields such as imaging.

As premilinary results, we show that one can calculate a condenser p-capacity by solving a p-Laplace equation with Dirichlet boundary condition and we provide asymptotic bounds to the p-capacity of any condenser of which the internal part has a non-empty interior. We provide the asymptotic expansion when $p = d$ and for a ball-shaped compact when $p < d$. We then directly establish the positivity rule for the condenser capacity of a point. When the condenser capacity of a point is positive ($p > d$), we estimate the speed of descent of the p-capacities of balls down to that of a point.

Then our main contribution is to establish the thorough positivity rule for condenser p-capacity of segments by introducing so-called equidistant condensers. This new method brings up the meaningful relationship existing between the capacity of a segment in a d-dimensional domain with the capacity of a point in a d-dimensional domain and more significantly with the capacity of a point in a $(d-1)$-dimensional domain. This result paves the way to induction reasonings for proving positivity rules of condenser capacities of k-dimensional compact submanifolds of higher dimensions.

When the condenser capacity of a segment is positive ($p > d - 1$), we introduce so-called elliptical condensers for estimation purposes. In the linear case $p = d = 2$, we provide a sharp estimate of the 2-capacity of a segment along with the asymtotic expansion when the length of the segment tends towards zero. We then briefly discuss why elliptical condensers might help computing capacities of segments.

While various types of capacities are commonly used to study the local behaviour of solutions to quasilinear elliptic equations, far less is known about condenser capacities themselves. Many questions remain to be studied, both on the theoretical side, such as estimates of the speed of descent in property (iv) of Theorem 1.2, and on the numerical one, such as ways for easily computing condenser capacities. As a result, the ultimate goal will be to develop methods allowing efficient use of condenser capacities in applicative tasks requiring automatic detection and quantification of zero measure sets.

6. Bibliography

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